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AN AUTOSOMAL RECESSIVE FACTOR INDUCING SEMI-STERILITY IN DROSOPHILA MELANOGASTER FEMALES

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Measurements of the frequency of "dominant lethals" induced in sperm of Drosophila[1] have been seriously hampered by appreciable uncontrolled discrepancies between the results of parallel experiments. It was found that when untreated material was used in mass matings, the number of eggs that failed to develop into adult flies showed non-random variations from zero up to about 40%, averaging about 20%. The failures in development such as were observed in eggs from untreated parents are consequently superimposed on the induced failures to be measured in experiments on "dominant lethals." Reducing the non-random failures in development of untreated material to a negligible amount should, therefore, greatly improve the accuracy of experiments on "dominant lethals." With this in mind, an effort was made to investigate the causes for failures in development of untreated material. The results are reported in this paper.

Experimental Methods.—The methods used varied somewhat in different experiments. Eggs were collected from single females. Failures of development were observed by counting either the relative numbers of hatched and unhatched eggs or the number of adult flies emerging per 100 eggs collected. In the former case the most satisfactory procedure was to collect eggs which had been deposited on corn meal agar (coated with fresh yeast) spread on a wooden spoon. The spoons were then stored for incubation in empty vials, proper care being taken to ensure sufficient moisture. In the latter case it proved advisable to transfer the eggs to a corn meal agar slant within a vial, in order to minimize the chances of accidental deaths. When only the presence or absence of a high sterility effect had to be determined (see further below), it was sufficient to collect eggs on corn meal agar within a vial and to examine the vial after incubation to see whether most of the eggs had or had not hatched.

Results.—Preliminary observation of the failures in development of untreated material showed the following points: (*a*) The phenomenon can already be observed at the egg-hatching stage. (*b*) A large share of the failures in development occurs among the offspring of comparatively few females ("semi-sterile" females); the non-random occurrence of unhatched eggs in mass cultures is thus related to the variable number of eggs laid by "semi-sterile" females within any one experimental sample. (*c*) The frequency of failures in development among the offspring of "semi-sterile" females ranges anywhere up to 100%; successive tests on eggs laid by the same female yield comparable results, indicating that the failures are due to biological causes and not to experimental accidents. (*d*) "Semi-sterile" females do not seem to be confined to the inbred wild type *Swedish b* stock on which most of the experiments were made. (*e*) "Semi-sterility" is not affected when a female is outbred to a male from a different stock, and therefore it appears to be an actual characteristic of individual females.

Dr. B. P. Sonnenblick kindly supplied a different *Ore R* stock, which was selected several years ago at New York University for high egg-hatchability and which will be referred to as *Ore R* (*S*). The failure of egg-hatching is much less frequent in *Ore R* (*S*) than in other stocks, and averages only about 4%. When eggs are collected from single females of this stock, the failures of egg-hatching among the offspring of different females are, however, still non-random. Successive egg collections from the same female seem to indicate that in this case also a lower rate of hatching is not accidental but related to biological individual factors. Failures of egg-hatching, however, only rarely exceed 15% among the offspring of any individual female.

The following reciprocal matings were then carried out: (*a*) *Ore R* (*S*) ♀ by *Ore R* (*S*) ♂, (*b*) *Ore R* (*S*) ♀ by *Sw b* ♂, (*c*) *Sw b* ♀ by *Sw b* ♂, (*d*) *Sw b* ♀ by *Ore R* (*S*) ♂. After one day, 15 females were isolated from each mating, and hatchability was tested on two successive samples of eggs from each female. (Out of 60 tests, 2 were discarded because no egg was laid, and 1, belonging to group (*c*), because all eggs lacked chorion and failed to hatch.) The observed numbers of hatched and unhatched eggs for the different groups were: (*a*) 857:49, (*b*) 692:26, (*c*) 868:129, (*d*) 732:179. Failures of hatching never exceeded 25% for any one female of groups (*a*) or (*b*), but exceeded 50% for one female of group (*c*) and for three females of group (*d*). This suggests that the non-random failures of hatching are largely determined by the female parent.

The eggs laid by one of the "semi-sterile" females of group (*d*) were allowed to develop and to undergo free brother-sister mating. Eggs were then collected from 11 single F_1 females and tested. The total numbers of hatched and unhatched eggs were 843 and 6. The conditions which were responsible for the "semi-sterility" of the *Sw b* parental female thus ap-

peared to be wholly recessive, if inheritable at all. (However, later tests on the offspring of several F_1 females, hybrids of *Ore R* (*S*) and *Sw b*, showed that the incidence of failures in egg-hatching is not lower, on the average, for these hybrid females than for inbred *Ore R* (*S*) females.)

Six strains of F_2, arising from as many F_1 females out of the group of 11 referred to above, were allowed to develop and were mated by brother-sister pairs. Approximately 20 females from each strain were tested for egg-hatchability. Eggs were collected for one day from 117 females; 8 tests were discarded because no eggs or only 1 or 2 eggs were collected, while all other females laid at least 10 eggs. The results are shown in table 1. The frequency distribution in the last row is clearly bimodal, indicating segregation of some major factor. If 50% is taken as the abscissa separating the two modes, the total frequencies within each mode are 82 and 27; that is, they are approximately in the ratio 3:1, indicating the probable existence in the parental female of a single major mendelian factor. The frequency distribution is, however, not quite homogeneous among different strains; this should be related to minor factors which were heterozygous in the parental generation.

TABLE 1

TESTS OF THE FERTILITY OF THE F_2 OF A SINGLE "SEMI-STERILE" *Sw b P* ♀

	NUMBER OF FEMALES CLASSIFIED ACCORDING TO THEIR FERTILITY													
	PERCENTAGE OF UNHATCHED EGGS													
STRAIN NO.	0	0 TO 10	10 TO 20	20 TO 30	30 TO 40	40 TO 50	50 TO 60	60 TO 70	70 TO 80	80 TO 90	90 TO 100	TOTAL NUMBER OF ♀'S TESTED	TOTAL EGGS HATCHED	NUMBER OF OBSERVED UNHATCHED
1	1	5	1	..	1	..	1	1	2	2	..	14	290	150
2	5	10	1	1	..	1	..	1	..	...	..	19	723	89
5	10	7	..	1	..	..	..	..	1	..	1	20	579	81
6	6	6	1	2	1	1	..	1	..	1	1	20	563	182
7	6	5	..	..	..	..	2	3	2	1	..	19	428	181
8	3	7	..	..	..	..	1	4	1	..	1	17	450	150
Total	31	40	3	4	2	2	4	10	6	4	3	109	3033	833

The F_3 from the 27 "semi-sterile" F_2 females which had shown over 50% failure of egg-hatching was further tested for fertility. This was actually a test to determine which F_2 males had been homozygous for the same major factor as the females to whom they were mated. Beginning at this stage of the experiments, the classification of a female as "semi-sterile" was arbitrarily determined according to whether the percentage of failures in egg-hatching did or did not exceed 50%. In the large majority of cases an actual egg-count was not carried out, and the classification could easily be performed by glancing at a batch of eggs after incubation. All females within 6 of the 27 strains of F_3 appeared to be "semi-sterile." These 6 strains were carried further; inbreeding of each strain and cross-breeding between strains produced only "semi-sterile" females, indicating

that a single "semi-sterility" factor—thereafter tentatively called *sst*—had been isolated.

The segregation experiments reported above made it appear likely that *sst* is autosomal. The usual test for location of a gene was then carried out: *sst/sst* males were mated to *Pm/Cy H/Sb* virgin females, and 10 F_1 *Cy Sb* males were back-crossed singly to *sst/sst* virgin females. The phenotypically different classes of F_2 females were tested for *sst*, with the following results: of 72 non-Curly females tested, 57 produced offspring and all were homozygous for *sst;* 95 out of 107 tests on *Cy* females were successful, and all but 2 of them were clearly not "semi-sterile." *Sb* appeared to be wholly unrelated to *sst*. Therefore, *sst* is located in the second chromosome. The nature of the exceptions represented by the two "semi-sterile" *Cy* females is still under investigation. These flies were the offspring of two different F_1 males.

Discussion.—The constitutional characteristics of a female affecting the viability of its offspring might have been expected to be determined by the collective action of a large number of factors, each of them only slightly important in itself. The probable identification of the factor *sst* indicates that this is not so.

Several questions which may be of some interest, particularly to the physiological geneticists, can now be considered. First, the way in which *sst* acts can be investigated. The failures of development in the offspring of *sst* females probably occur at an early stage of the egg development because the unhatched eggs do not show any obvious sign of embryonic development when dechorionated. It is possible either that the eggs themselves are defective or that *sst* females do not succeed in utilizing properly the sperm they have received.

Further investigation might also be directed toward determining how widespread the factor *sst* is among several laboratory stocks. It is not yet clear how this factor survived throughout a long series of experiments prior to its identification, in spite of its great selective disadvantage.

Finally, the problem of controlling the failures of development is not completely solved by eliminating *sst*. Females other than *sst*, whose offspring show embryological disturbances to a high degree, have frequently been found. Random presence of such females in any experimental sample is liable to have a disturbing effect. The impression has been gathered that many more genetic factors inducing failures of development to a more or less striking degree could be easily identified. Investigation of such factors would be especially interesting if they appeared to act through the production of defective eggs, lacking, for instance, in some growth factor.

The "semi-sterility" of the two exceptional *Cy* females may have been due to factors other than *sst*, although the ratios of segregation (1:17 and 1:11) appear to be rather small. If, however, the factor *sst* was actually

present in the same chromosome as *Cy*, it ought to have arisen by mutation, since no crossing-over occurs in males.

Summary.—A wild type strain, most of whose eggs fail to develop, has been isolated. Breeding experiments indicate that this behavior is due to a second-chromosome factor which affects the offspring of homozygous parent females independently of the genetic constitution of their mates.

[1] Fano, U., and Demerec, M., *Genetics*, **26**, 151 (1941).

"ATAXIA," A HEREDITARY NERVOUS DISORDER OF THE RABBIT

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Many degenerative disorders of the nervous system in man occur more often in certain families than in the general population, suggesting that inheritance is involved, but conclusive evidence for this is difficult to obtain.

But in laboratory animals produced in large numbers under uniform conditions and inbred as closely as is practicable, evidence as to the inheritance of nervous disorders is readily obtained. We have observed such a case in the rabbit colony at Brown University, in a family of chinchilla rabbits which has been under continuous observation for nearly 15 years, and in which parent-offspring inbreeding had been practiced for at least 5 generations at the time the disorder was first discovered. The disorder is one which for lack of a more precise descriptive name we have called "ataxia." It is due to a semi-lethal mutation which is inherited as a simple recessive character. The genetic symbol to be used in reference to this mutation is *at*, its normal and dominant allele being *At*.

The first affected animal was observed late in the winter of 1936, in a litter of six, all of which appeared vigorous and entirely normal at birth. At two months of age one of the females became paralyzed in the hind quarters. She was kept under observation but died after a period of two weeks during which time the condition became progressively worse. Each of the other two females of this litter remained entirely normal, was raised to maturity and mated back to the father. Each produced young in her first litter which showed an identical condition setting in at approximately the same age. This gave rise to the suspicion that the defect might be due to a hereditary mutation.

Since affected individuals do not survive until sexually mature, all which occur are born of normal parents. Accordingly the mutation must be re-

cessive in inheritance. When it arose cannot be stated since both parents of the first affected individual must have carried the mutant gene derived from some common ancestor.

The first ten individuals which manifested the defect were progeny of the same male (4207), a vigorous normal rabbit still sexually active although seven years old. He produced "ataxic" young by two females of quite different immediate ancestry, one belonging to his own line of descent, the other derived from an outcross of his ancestor (♂48D3) with an entirely different line of chinchilla rabbits. Accordingly the mutation must have been already in existence as a recessive in the nearest common ancestor of both females, viz., ♂48D3.

The defect manifests itself first in the hind legs as a disinclination to movement and secondly in the manner of movement which is forced and irregular. Instead of the characteristic hopping movements of normal rabbits with both hind legs working together, there is a tendency for them to be used alternately and with seeming lack of coördination. In the next stage one of the hind legs appears to be stiffened and extended to one side. The weight of the body is then carried by the other leg, producing a posture which is quite characteristic. Even before this can be readily recognized a tendency may be noted for the animal to come to rest close to the side of the cage, as if for support.

The gait becomes progressively clumsier, the legs gradually lose their ability to maintain support of the body and the animal, after walking a short distance, may fall over on its side in an apparently helpless condition. After a rest, it may be able again to utilize its legs in support of the body. In from 5 to 10 days the condition may progress to the point where the animal is almost completely incapacitated. At this stage the hind legs can never be used to support the body. The animal lies upon its side, usually the one opposite the leg which first became affected, with the hind legs extended. When the animal is disturbed, walking movements may be attempted but exhaustion comes rather quickly.

At this stage the animal seems hypersensitive to many situations which are not disturbing to a normal rabbit. Although sudden movements and loud noises do not cause violent reactions, the animal does appear momentarily startled. Continued stimulation such as caused by people moving about the room leads to restlessness which may continue long after the stimulus has been removed. This hypersensitivity is also strikingly manifested in the efforts of the animal to drink from the customary brass nipple and water bottle employed in our laboratory. The vibrissae appear so highly sensitive that any approach to the nipple causes violent withdrawal and may account in part at least for the very apparent resultant dehydration of the animals at autopsy. It is also interesting to note that the animal appears less sensitive to glass than to brass nipples. At about

this time a characteristic grinding of the teeth, which continues at irregular intervals until death, becomes apparent.

The appetite appears to differ with the individual. Some animals lose appetite very soon after onset of the condition. Others have shown a desire for food for some time after they were no longer able to gain their feet, and therefore had to be fed by hand. Food is located with difficulty even if placed in close proximity to the head. In general it has been found that succulent green food will be taken long after all other food is refused and a better general condition is maintained as a consequence.

Sensitivity and excitability seem to increase to a definite peak which varies with the individual. The patellar reflex first shows movements of extremely low magnitude followed by increasing rapidity and magnitude until a crossed reflex is present. At no time does the patellar reflex appear pendular. This reflex degeneration appears to progress anteriorly until the fore legs and head are involved. A crossed reflex has never been observed in the fore leg although responses of low magnitude result followed by responses of high magnitude. At this stage some individuals show a pronounced lateral nystagmus. Breathing tends to be convulsive and orientation of head movements become difficult. Pain reflexes continue to be present until death, although some minor changes in them have been observed.

In the final stage, the animal becomes more or less flaccid, and in contrast to the preceding hypersensitivity, becomes relatively quiescent with a marked drop in body temperature. It is not easily stimulated and apparently is unaware of its surroundings. All desire for food or water is lost. Respiration is more irregular and labored and death follows within a few hours.

Post-mortem examination usually shows excessive dehydration and emaciation even in those individuals having the best appetites. The tongue shows no abnormality such as is described for vitamin B deficiencies, with which certain types of paralysis are sometimes associated, nor is the coat in any way affected. The teeth, particularly the incisors, are frequently worn and chipped beyond those of normal sibs, which is not surprising in view of the continuous grinding already mentioned. The adrenals in those animals which have been examined seem to be smaller than is normal for rabbits of the same age although we are not able to attach any particular significance to this observation. Other post-mortem irregularities have not been observed.

The time of first manifestation of the condition has varied from 58 to 89 days after birth, the mean for 41 individuals being 72.9 days. The true mean may be earlier since the earliest indications of onset could easily be overlooked. The sequence of the above described events has been the same in all cases observed, each stage differing only in duration and inten-

sity. The longest duration of the condition was observed in a case lasting 27 days, death probably being due to involvement of the respiratory centers. Several have succumbed five days after first manifesting symptoms. Preliminary microscopic examination of the nervous system has revealed demyelinated areas in the medulla in the general vicinity of the vestibular nuclei. Although we cannot as yet state the exact extent or variability of the lesions, it seems probable that vestibulo-cerebellar and ponto-cerebellar systems are involved. These observations as well as the symptoms and reflex behavior described above indicate that we are dealing with an extrapyramidal degenerative disorder.

Inasmuch as the affected rabbits are always derived from entirely normal appearing parents, and die before reaching sexual maturity, the character may be described as a sublethal recessive. It can be perpetuated and studied only by way of the heterozygous carriers. Matings between such known heterozygous males and females have produced a total of 219 young which have been reared to at least three months of age. The affected individuals number 55, the expectation for a monofactorial recessive being 54.75; hence there is no prenatal lethality of the mutation.

The use of extracted recessives in the customary outcross, back-cross and *inter se* recessive matings being impossible, known heterozygotes were used in their stead. In 12 out-cross matings involving two different unrelated breeds, family X descended from Castle's small race and family III a New Zealand White, 64 normal young have been obtained. The subsequent back-cross matings of these F_1s to known heterozygotes of family V appear to fall in two categories, those which produce affected young and those which do not. Six or more normal young have been regarded as an adequate test. Of such tested F_1s, 6 were heterozygous (*At at*) and 4 were homozygous (*At At*), equality being expected.

Of the normal individuals derived from heterozygous parents, $^1/_3$ should be homozygous and therefore incapable of producing affected young, whereas $^2/_3$ should be heterozygous and so capable of producing affected young. Of 34 individuals tested, 12 proved to be homozygous and 22 heterozygous, which is close to the expected 1:2 ratio. Affected individuals occur in both sexes without statistically significant difference in frequency.

As in the case among pigeons described by Riddle[1] it is not certain by what name this disorder should be called. In the sense that it represents a loss of coördination of movements it is an ataxia. However, neither in the symptoms manifested nor in the pathology described does the case of the pigeon seem to be strictly comparable with this. This disorder is more closely related to the condition described in the rabbit by Nachtsheim[2] as spastic spinal paralysis. But the time of onset of the present defect is much later than that reported by Nachtsheim and the duration is shorter. The pathological disturbance appears to be located in the same general re-

gions of the brain although it is difficult to be certain as to their location from the information available for the spastic paralysis. The inheritance in all of these cases is similar. The case of paralysis in the dog described by Stockard[3] is entirely dissimilar in symptomatology, pathology and also in inheritance.

The present disorder, due to its semi-lethal nature, is not ideal genetic material with which to work but genetic tests of the possible chromosomal association with other known genes have been devised which at the same time are providing an economical and continuous supply of affected animals with which to test the functional and organic nature of this condition, which so closely parallels certain human degenerations.

Conclusion.—We may conclude that this degenerative disorder involving the brain stem, which appears to have arisen by a mutation, is transmitted as a simple mendelizing unit. By crossing it can be introduced into unrelated and unaffected families to become manifest in subsequent generations. Likewise it can be eliminated by proper breeding tests accompanied by selection of those segregates which are free from the defect, for establishment of a new line.[4]

[1] Riddle, O., *Proc. Soc. Exper. Med. and Biol.*, **15**, 56 (1918).
[2] Nachtsheim, H., *Deut. Tierärztl. Wochenschrift*, **44**, 742–746 (1936).
[3] Stockard, C. R., *Am. Jour. Anat.*, **59**, 1–54 (1936).
[4] This study is a portion of a research program supported in part by a grant from the Rockefeller Foundation.

THE IDENTIFICATION AND CHARACTERIZATION OF BACTERIOPHAGES WITH THE ELECTRON MICROSCOPE

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Bacteriophages, or bacterial viruses, are a group of viruses reproducing in the presence of living bacterial cells. Bacteriophages are particulate, and convincing evidence exists that (1) one particle of phage is sufficient to originate the lysis of a bacterial cell; in the lysis, a variable number of new phage particles (an average of 100 or more) are liberated per cell;[1] (2) the elementary particles of each phage strain seem to have a characteristic particle size as determined by any one of various indirect methods of investigation (ultrafiltration,[2] radiosensitivity,[3] diffusion,[4]) and diameters ranging from 10 to 100 mμ have been obtained for the various strains de-

pending on the method of investigation, although diffusion experiments occasionally yield still smaller values.

The electron microscope has recently been applied with success to the study of viruses[5] and it therefore seemed desirable to attempt such a study of bacterial viruses, particularly since they offer favorable possibilities for the identification of the virus particles through a study of the reaction between the individual particles and the bacterial cell under the microscope. Indeed, a number of short reports have been published recently by German authors[6, 7] in which round particles have been described as corresponding to the phage particles, although Ruska[7] shows pictures of "sperm-shaped" particles from a phage suspension adhering to a bacterial membrane. From this evidence alone he is unable to decide whether these are particles of phage or bacterial products.

We have undertaken an investigation of the problems of phage structure, size, reproduction and lytic activity by means of the RCA electron microscope. Research on the last items is still in progress. The present report concerns itself with the identification and the morphological analysis of a number of strains of phage particles and their adsorption on sensitive bacterial cells. The results are illustrated by some of the electron micrographs (Plates I and II) which have brought to light many extremely interesting features. Details of the material and methods used will soon be published.

I. Bacteriophage anti-coli PC (particle diameter by diffusion 44 mγ, Kalmanson and Bronfenbrenner[8]; by x-irradiation 50 mμ, Luria and Exner, unpublished).

Micrographs of high titer suspensions, figures 1, 2, 4, 5 and 6, show the constant presence of particles of extremely constant and characteristic aspect. They consist of a round "head," and a much thinner "tail," which gives them a peculiar sperm-like appearance. The "head" is not homogeneous but shows an internal structure consisting of a pattern of granules, distinguished by their higher electron scattering power. Deviations from the usual symmetrical internal pattern may be due to varying orientation of the particles or to other factors as yet unknown. The diameter of the head appears to be about 80 mμ; the tail is about 130 mμ long.

EXPLANATION OF PLATE

PLATE I

1. Electron micrograph of particles from a high titer suspension of bacteriophage anti-coli PC. × 38,000.
2. Particles from a high titer suspension of bacteriophage anti-coli PC. × 84,000.
3. *Escherichia coli* from suspension in distilled water. × 17,000.
4. *Escherichia coli* in suspension of bacteriophage anti-coli PC for ten minutes. × 17,500.

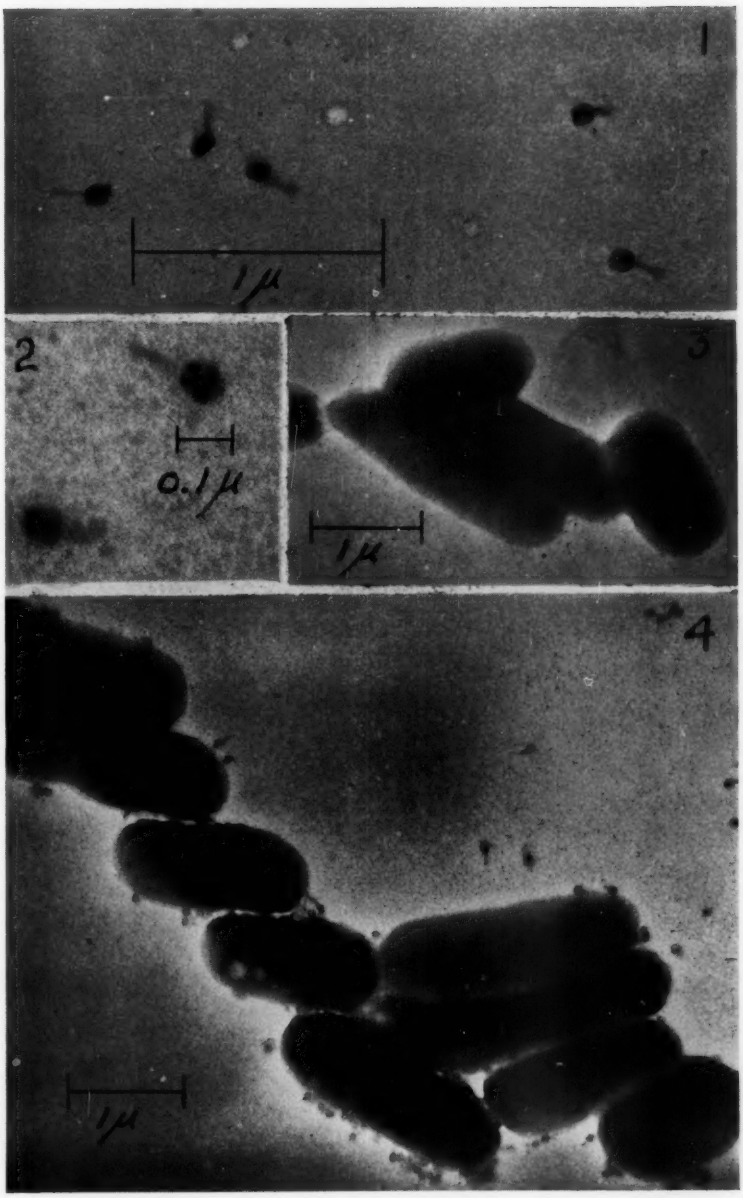

PLATE I

This gives a size which is in fair agreement with the figures deduced from the radiosensitivity method. On the other hand, it is possible that the size as determined by x-rays corresponds more closely to the size of the granules.

When allowed to stand a few minutes in the presence of sensitive bacterial cells *Escherichia coli*, strain PC (Fig. 3), the particles described above are readily adsorbed (Figs. 4 and 5). They appear to stick to the bacteria either by the head or by the tail. Other conditions remaining constant, the number of particles adsorbed on a bacterium increases with the time of contact, although it is difficult at the present time to differentiate between adsorption and reproduction of the particles on the cell wall. By allowing the phage to stay in contact with bacteria for a time of the order of the minimum time of lysis (21 minutes for PC phage, Delbrück and Luria[1]) it is possible to observe bacterial cells extensively damaged, surrounded by a very large number of particles, probably newly formed (Fig. 6).

II. Bacteriophage anti-coli P 28, also active on *Escherichia coli* strain PC (particle size: irradiation, 36 mμ, Luria and Exner.[3]

Round particles are visible in the suspensions of this phage which are somewhat smaller than those described for phage PC (about 50 mμ in diameter). An extremely thin tail, although difficult to demonstrate with certainty in the reproductions, seems to be visible in many instances. In many micrographs the head is almost completely filled by a dense internal structure. These particles, too, are readily adsorbed on sensitive bacterial cells.

III. Bacteriophage anti-staphylococcus 3K (particle size: by ultrafiltration and ultracentrifugation 50–75 mμ, Elford;[2] by irradiation 48 mμ, Luria and Exner.[3]

Owing to technical reasons, the conditions for successful micrographing are here less favorable. Nevertheless, the presence of approximately round particles of proper size has been established in preparations of this page also.

We are inclined to identify the particles described above with the actual particles of bacteriophage for the following reasons: (*a*) They are always present in highly active phage suspensions and missing in any control suspensions (media, bacterial cultures, bacterial filtrates, etc.); (*b*) they are readily adsorbed by the bacterial cells of the susceptible strain and fail to be adsorbed by other bacteria; (*c*) the size from a given strain is uniform and corresponds essentially to measurements by indirect methods; (*d*) the structure of both the "head" and the "tail" is characteristic of the strain of phage; (*e*) preliminary experiments on the lysis process seem to demonstrate the liberation of these particles from the lysing bacteria.

Conclusions.—We do not want to discuss here the bearing of the above described results on the problem of the nature of bacteriophage and of viruses in general. We limit ourselves to pointing out the extreme interest of the finding of such constant and relatively elaborate structural differen-

tiation in objects of supposedly macromolecular nature. This result is of equal interest in the field of genetics, since genes, together with viruses, are currently supposed to be macromolecular entities.

The correspondence of the particle size as directly portrayed in the electron microscope with the results of indirect methods is also very remarkable. although it does not exclude the possibility of phage activity being sometimes associated with smaller particles. It is worth while emphasizing that the results of the present investigation, together with the recently published results of irradiation of bacteriophages, represent most desirable evidence for the validity of the so-called "hit theory" for the determination of the "sensitive volume" in sub-light-microscopic biological objects. This conclusion, too, seems to be interesting from the point of view of genetics, since the "hit theory," although widely criticized, has been used for calculating the approximate value of the dimensions of genes.

The authors are grateful to the National Research Council Committee on Biological Applications of the Electron Microscope for allocating time for this study, and to the RCA Laboratories for the use of their facilities, and to Dr. V. K. Zworykin for his interest and encouragement. The authors also thank Dr. Stuart Mudd for the use of the facilities of his laboratory for the preparation of material for study.

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† RCA Fellow of the National Research Council.

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[4] Hetler, D. M., and Bronfenbrenner, J., *Jour. Gen. Physiol.*, **14**, 547 (1931).

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[7] Ruska, H., *Naturwiss.*, **29**, 367 (1941).

[8] Kalmanson, G. M., and Bronfenbrenner, Jr., *Jour. Gen. Physiol.*, **23**, 203 (1939).

EXPLANATION OF PLATE

PLATE II

5. *Escherichia coli* in suspension of bacteriophage anti-coli PC for 20 minutes. × 14,500.

6. *Escherichia coli* in suspension of bacteriophage anti-coli PC for 20 minutes. × 12,500.

7 and 8. Particles from a high titer suspension of bacteriophage anti-coli P28. × 38,000.

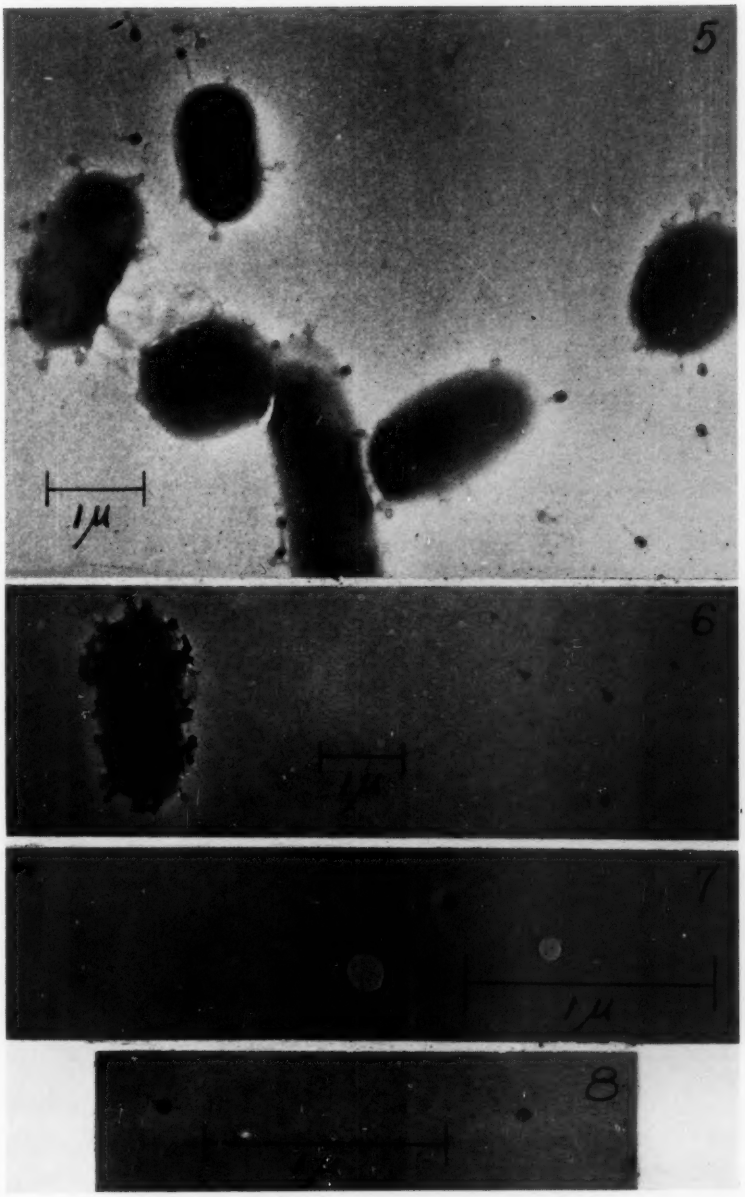

PLATE II

CONCENTRATION OF GROWTH HORMONE AND FRUITFULNESS IN THE MONTMORENCY CHERRY

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Several years ago the writer[2] discovered that varieties of oranges, lemons and grapes that produce seedless fruits have a higher concentration of growth hormone in the ovary of the flower than varieties that produce seeded fruits. This led to the interesting speculation as to whether there might be any relation between growth hormone content of the ovaries and fruitfulness in a species of plant. But no immediate material was available.

A visit to Prof. V. R. Gardner, Director of the Michigan State Agricultural Experiment Station, brought forth the information that the Station had selections of fruitful and barren Montmorency cherries. These trees were put at the disposal of the writer, and it was decided to set up an experiment to investigate the growth hormone content of these selections. This investigation was continued for three seasons. Three trees of a high yielding selection and three trees of the low yielding selection were chosen. The high yielding trees were nineteen years old and of a standard type; the bearing records of these trees were available for the whole period. The low yielding selections originated as a bud sport about 1923.[1] The trees chosen were nine years old; their yield record, and that of the tree from which they had been derived, was also known. For several years the percentage of setting had been determined for these various trees, and it was found that on the barren trees from 0.4 to 11.0 per cent of the blossoms produced fruits, whereas the fruitful trees set from 20.0 to 47.0%. These figures show that there is a great difference in the percentage of setting in the two lots of trees. Gardner[1] has examined numerous ovaries from the barren trees, and has found that they are perfectly normal. He states that the flowers are somewhat larger than those on the fruitful trees and the writer has noticed that too, as well as that the ovaries are larger. Every year that the trees have been examined, it was observed that the barren trees produced a profusion of blossoms.

The procedure followed in the experiment was to collect blossom buds that were ready to open or were in the process of opening. By two cuts, the ovary was separated from the bud. The ovaries were placed in closed vials until a number had been obtained, and during the first two seasons these ovaries were placed in freshly distilled ether as soon as weighed, but in 1941 they were frozen with dry ice as soon as dissected out, and were kept frozen until reaching Ann Arbor, where they were boiled for one minute

before placing in ether.[3] The further extraction and auxin assay were made in the usual way.

Material for the first analysis was collected on May 9 and 10, 1939. Three different collections of flowers were made during these two days, and each time flowers were gathered from both groups of trees so that there would be no difference in the weather at the time of collection. As shown by table 1, the growth hormone concentration was approximately 23% greater in the ovaries from the fruitful trees than from the barren trees. In 1940, the flowers were collected from the barren trees in the afternoon and evening of a very warm and sunny day, and collections were made from the fruitful trees next morning and afternoon during a cold rain. This time the barren trees showed a slightly higher growth hormone content.

TABLE 1

RELATION BETWEEN AUXIN CONCENTRATION IN THE OVARIES AND FRUITFULNESS IN THE MONTMORENCY CHERRY. AUXIN CONCENTRATION IS EXPRESSED AS INDOLE ACETIC ACID EQUIVALENTS IN GAMMAS PER KILOGRAM OF FRESH MATERIAL

	FRUITFUL TREES			UNFRUITFUL TREES		
YEAR	AMOUNT OF MATERIAL, IN GRAMS	AUXIN CONCENTRATION GAMMAS PER KILOGRAM	YIELD OF FRUITS, IN POUNDS	AMOUNT OF MATERIAL, IN GRAMS	AUXIN CONCENTRATION GAMMAS PER KILOGRAM	YIELD OF FRUITS, IN POUNDS
1939	3.445	20.07	770	2.390	16.27	2.12
1940	9.760	16.57	321	4.125	17.92	Only a few fruits
1941	4.456	24.90	430	3.390	8.02	61.0

NOTE: Three trees were used of each kind.

The results for the two years left the matter in a very uncertain situation. While some unpublished data seem to indicate that growth hormone content is higher in a plant in a warm temperature than in a cold one, the information is not too positive. If that were the situation, we might be able to explain the low concentration in the fruitful trees and the high one in the barren trees. To avoid any such complications, collections were made from all trees at the same time in 1941. Both days that the material was gathered this year the weather was very warm and the days were sunny. This year, the fruitful trees had a growth hormone content more than three times as great as that of the barren trees. The experiment in 1941 was the most carefully executed of all. The new method[3] of freezing and destroying the enzyme system by boiling has been found to give results more indicative of the true values of active growth hormone in a plant than the older method. This year greater care was also exercised in the selection of the flower buds, and in the time of collection.

The results seem to the writer to justify the conclusion that in the Montmorency cherry there is a positive correlation between fruit setting and the growth hormone content of the ovary of the flower. But before

one can draw any general conclusions, more comparisons between barren and fruitful plants would have to be made.

The writer wishes to express his gratitude to Professor V. R. Gardner for his aid in this investigation, and also to the Directors of the Horace H. Rackham Research Fund for financial aid.

[1] Gardner, V. R., "A Study of Some Unproductive Sports of the Montmorency Cherry," *Jour. Agri. Res.*, **50**, 457–478 (1935).

[2] Gustafson, F. G., "The Cause of Natural Parthenocarpy," *Amer. Jour. Bot.*, **26**, 135–138 (1939).

[3] Gustafson, F. G., "The Extraction of Growth Hormones from Plants," *Ibid.*, **28**, 947–951 (1941).

THE CALCULATION OF ROTATION FACTORS FOR ECLIPSING BINARY SYSTEMS

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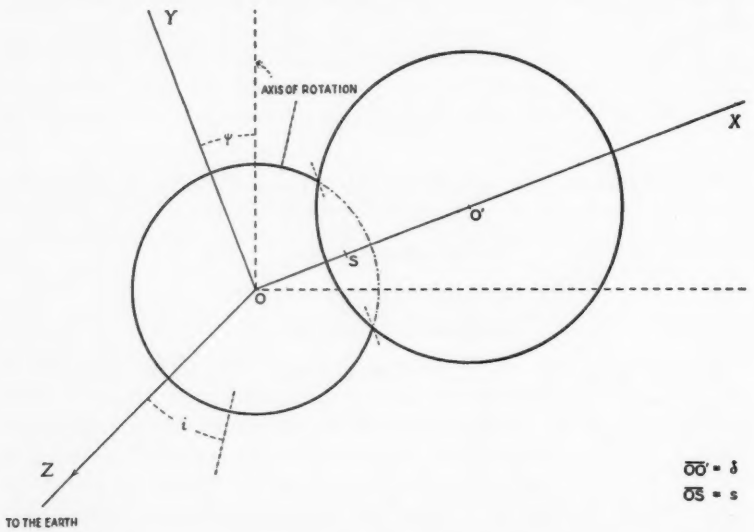
The interpretation of the rotational effect, exhibited by the velocity curves of eclipsing systems within minima, appears to be one of the few possible methods to specify the absolute dimensions of single-spectrum binaries. A first step of such a procedure is to determine the velocity of rotation V_e at the equator of the star undergoing eclipse. If R denotes the observed rotational effect, i.e., the deviation of observed radial velocity of the eclipsed star from a simple elliptic motion in a plane inclined to the celestial sphere by an angle i, then the ratio

$$R/V_e \sin i = F \qquad (1)$$

defines a dimensionless quantity which depends only on the geometry of the eclipse and on the distribution of brightness over the apparent disc of the eclipsed star. The quantity F defined by the foregoing relation is known as the rotation factor. Its value varies with the phase, but can evidently be predicted for any particular moment provided that the geometrical elements of the eclipse are known from the study of a light curve. The aim of the present paper will be to establish analytical formulae expressing F explicitly in terms of such elements for stars which appear uniformly bright or are darkened at limb to any arbitrary degree. Formulae relevant to uniform discs have already been set up by Petrie;[1] but his results will be found to admit of considerable simplification; while the darkened case has so far not been tackled analytically at all.

First, let us define F in terms of the geometry of eclipse. Provided that the star rotates like a rigid body, the radial velocity at any point on the apparent disc—apart from any orbital motion—is proportional to the distance l of that point from the projected axis of rotation. Let us refer our eclipsing system to a rectangular frame xyz (see the figure), with origin at the center of component undergoing eclipse, defined so that the xy-plane is tangent to the celestial sphere and the z-axis coincides with the line



of sight, while the x-axis is constantly in the direction of the projected center of the secondary component. Then, evidently,

$$F = \frac{\int\int J l dx dy}{r_1 \int\int J dx dy}, \tag{2}$$

where r_1 is the radius of the star undergoing eclipse and J specifies the distribution of brightness over its apparent disc. The limits of integration are to be extended over the whole area visible at any particular phase.

Let, further, θ denote the longitude in the plane of a circular orbit, reckoned from the mid-primary minimum, and δ be the apparent separation of centers of both components expressed in terms of the orbital radius taken as unit. The distance l of a point, on the apparent disc, from the projected axis of rotation then becomes

$$l = x \cos \psi - y \sin \psi, \tag{3}$$

where ψ, the angle between the axis of rotation projected on the celestial sphere and the y-axis, follows from

$$\delta \cos \psi = \sin \theta \quad \text{and} \quad \delta \sin \psi = \cos \theta \cos i,$$

while

$$\delta^2 = \sin^2 \theta \sin^2 i + \cos^2 i.$$

As to J, the distribution of brightness over the apparent discs of spherical stars is governed by limb-darkening only, and hence

$$J = H(1 - u + uz), \tag{4}$$

where H denotes the intensity of radiation parallel to the line of sight; u, the coefficient of limb-darkening; and z is the cosine of the angle of foreshortening which, in terms of the xyz-coördinates, is

$$r_1 z = \sqrt{r_1^2 - x^2 - y^2}.$$

The above formula for the rotation factor then takes the form

$$F = \frac{\cos \psi \int\int Jxdxdy - \sin \psi \int\int Jydxdy}{r_1 \int\int Jdxdy}. \tag{5}$$

Since, however,

$$\int\int Jydxdy$$

integrated over any area symmetrical with respect to the x-axis vanishes, equation (5) immediately reduces to

$$F = \frac{\sin \theta}{\delta r_1} \left\{ \frac{r_1(1-u)\int\int xdxdy + u\int\int x\sqrt{r_1^2 - x^2 - y^2}\, dxdy}{r_1(1-u)\int\int dxdy + u\int\int \sqrt{r_1^2 - x^2 - y^2}\, dxdy} \right\}. \tag{6}$$

In setting up the limits of integration, which has to be extended over the whole area visible at any particular phase, we find it convenient to perform the integration first over the whole disc, then over the eclipsed area, and subtract. The advantage of so doing rests with the fact that the expressions in the numerator on the right-hand side of (6) vanish if integrated over the whole disc, while integrals in the denominator reduce to πr_1^2 and $^2/_3 \pi r_1^3$, respectively. We are then left with integrals extending over the eclipsed area, of the form

$$\left\{ \int_s^{r_1} \int_{-\sqrt{r_1^2 - x^2}}^{+\sqrt{r_1^2 - x^2}} + \int_{\delta - r_2}^{s} \int_{-\sqrt{r_2^2 - (\delta - x)^2}}^{+\sqrt{r_2^2 - (\delta - x)^2}} \right\} x^m z^n dxdy = \pi r_1^{2+m+n} \alpha_n^m, \tag{7}$$

where r_2 denotes radius of the secondary (eclipsing) component, and

$$s = \frac{r_1^2 - r_2^2 + \delta^2}{2\delta}.$$

The dimensionless quantities α_n^m defined by equation (7) are evidently identical with the associated α-functions recently investigated by the writer[2] in connection with effects of gravity-darkening upon light curves of close eclipsing binaries. The rotation factors rewritten in terms of the associated α-functions take the simple forms

$$F = -\frac{\sin\theta}{\delta}\left\{\frac{(1-u)\alpha_0^1 + u\alpha_1^1}{(1-u)(1-\alpha_0^0) + u\,(\tfrac{2}{3} - \alpha_1^0)}\right\}, \qquad (8)$$

the negative sign being due to the fact that the formulae were set up for a phase after the mid-primary minimum when the rotational effect is negative.[3] For a corresponding phase preceding maximum eclipse the factors are numerically the same, but of opposite sign.

The evaluation of the associated α-functions for various values of m and n has recently been completed by the writer,[2] so that only the final forms of functions involved in equation (8) need be quoted here. If n is zero (or an even integer) the respective associated α-functions can be expressed in terms of circular and algebraic functions. In particular, for $n = 0$ we have

$$\pi r_1^2 \alpha_0^0 = r_1^2 \cos^{-1}\frac{s}{r_1} + r_2^2 \cos^{-1}\frac{\delta - s}{r_2} - \delta\sqrt{r_1^2 - s^2}, \qquad (9.0)$$

and

$$\pi r_1^3 \alpha_0^1 = \delta r_2^2 \cos^{-1}\frac{\delta - s}{r_2} - \delta(\delta - s)\sqrt{r_1^2 - s^2}, \qquad (10.0)$$

if the eclipse is partial (regardless of whether the component undergoing eclipse is the larger or the smaller of the two), and

$$r_1^2 \alpha_0^0 = r_2^2, \qquad (9.1)$$

$$r_1^3 \alpha_0^1 = \delta r_2^2, \qquad (10.1)$$

if it is annular.

The functions α_0^0 and α_0^1 specify the rotation factors for uniformly bright stars alone.[4] If, however, there is some limb-darkening, two further associated α-functions with $n = 1$ ($m = 0$ and 1) are required for complete specification of our problem; and these are expressible only in terms of

elliptic integrals. Integrating (7) for the respective values of m and n, after some tedious algebra we obtain

$$\pi r_1^3 \alpha_1^0 = \frac{2}{3}\,\mathfrak{E} - \sqrt{\frac{\delta}{2}}\{{}_1^0\mathfrak{P}\omega_1 - \mathfrak{Q}_1^0\eta_1\}, \tag{11}$$

and

$$\pi r_1^y \alpha_1^1 = -\sqrt{\frac{\delta}{2}}\{\mathfrak{P}_1^1\omega_1 - \mathfrak{D}_1^1\eta_1\}. \tag{12}$$

In these equations

$$\frac{1}{2}\,\mathfrak{E} = \frac{\pi}{2} - \left\{E\left(\frac{\pi}{2}, \kappa\right) - F\left(\frac{\pi}{2}, \kappa\right)\right\} F(\phi, \kappa') - \left\{E(\phi, \kappa') - \frac{1}{2}\sqrt{\frac{\delta}{r_2}}\right\} F\left(\frac{\pi}{2}, \kappa\right),$$

$$\omega_1 = \frac{1}{\sqrt{e_1 - e_3}}\, F\left(\frac{\pi}{2}, \kappa\right), \quad \eta_1 = \sqrt{e_1 - e_3}\, E\left(\frac{\pi}{2}, \kappa\right) - e_1\omega_1,$$

where $F(1/2\pi, \kappa)$ and $E(1/2\pi, \kappa)$ denote the Legendre complete integrals of the first and second kind with the modulus

$$\kappa^2 = \frac{e_2 - e_3}{e_1 - e_3},$$

while the respective incomplete integrals possess an amplitude of

$$\phi = \sin^{-1}\sqrt{\frac{2\delta}{r_1 + r_2 + \delta}}$$

and a complementary modulus

$$\kappa'^2 = \frac{e_1 - e_2}{e_1 - e_3}.$$

The constants e_1, e_2, e_3, involved in the foregoing relations, are defined by

$$e_1 = \frac{1}{3}(\delta - s) + r_2,$$

$$e_2 = -\frac{2}{3}(\delta - s),$$

$$e_3 = \frac{1}{3}(\delta - s) - r_2,$$

if the eclipse is partial, and

$$e_1 = -\frac{2}{3}(\delta - s),$$

$$e_2 = \frac{1}{3}(\delta - s) + r_2,$$

$$e_3 = \frac{1}{3}(\delta - s) - r_2,$$

if it is annular.

The coefficients of ω_1 and η_1 in equations (11) and (12) take finally the forms

$$\mathfrak{P}_1^0 = \frac{2}{3}\left\{g_3 + \frac{1}{12}(2\delta - s)g_2 + h^2(h - 2s) - r_1^2(h - 4s)\right\},$$

$$\mathfrak{P}_1^1 = \frac{1}{2}\left\{\frac{1}{16}g_2^2 + \frac{1}{5}(8h - s)g_3 + \frac{1}{2}h(h - s)g_2 + (h^2 - r_1^2)(h^2 - 2hs + r_1^2)\right\},$$

$$\mathfrak{Q}_1^0 = \frac{4}{9}\left\{7r_2^2 - 4r_1^2 + \delta^2\right\},$$

$$\mathfrak{Q}_1^1 = \frac{1}{2}\left\{g_3 + \frac{1}{10}(14h - 3s)g_2 + 2(2h^3 - 3h^2s + r_1^2 s)\right\},$$

where $3h = 2\delta + s$, and

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3),$$

$$g_3 = +4e_1e_2e_3.$$

If the ratio of radii k and the geometrical depth of the eclipse p are defined, as usual, by

$$k = \frac{r_s}{r_l} \quad \text{and} \quad p = \delta - \frac{r_l}{r_s},$$

where r_l and r_s denote the radii of the larger and smaller component, respectively, the associated α-functions can evidently be made to depend on k and p alone and tabulated in terms of these parameters for any particular phase of the eclipse. In fact, for some of them, tables are already available. As the reader may easily verify,

$$\alpha_0^0 \equiv \alpha^U \quad \text{and} \quad \alpha_1^0 \equiv \frac{2}{3}\alpha^D,$$

where α^U is the fractional loss of light due to the eclipse of a uniformly bright disc (equal to the fractional area eclipsed), and α^D is a corresponding loss of light of a disc completely darkened at limb. Extensive and accurate tables of α^U and α^D have recently been completed by Zessewitsch.[5]

The denominators of (8) can therefore be found with the aid of tabular values with minimum of effort. No tables are, unfortunately, available so far for α_0^1 or α_1^1. It is to be hoped that such tables will be constructed in a due time; but at present their values will have to be computed for every case using the formulae given above. The calculation of α_0^1 is simple; that of α_1^1 is likely to be rather tedious—yet much less so than a graphical computation of the rotation factors for darkened stars would be.

The formulae for the rotation factors set up in the present paper are exact as far as the components can be regarded as spherical—i.e., as long as their dimensions expressed in terms of their separation are small. In close binaries, however, the forms of both components will sensibly deviate from a sphere on account of their axial rotation and mutual tidal action, and the distortion will also cause the surface brightness to vary proportionally to the gravity. In such cases the formulae developed in the present paper would not be rigorously applicable, but still should offer a close approximation.[6] Effects of distortion upon the rotation factors could be taken into account by modifying the limits of integration and adopting an appropriate form for $J(x, y, z)$, but such a procedure is likely to involve a very heavy algebra.

Another consideration to be borne in mind when applying the present results to practical cases is the probable failure of the assumption underlying equation (3), that the star undergoing eclipse rotates like a rigid body. This is indeed not the case for the sun where the equatorial regions rotate by about 10 per cent faster than regions near the poles, and is still less likely to be true for rapidly rotating components of eclipsing binaries.[7] The equatorial velocity obtained with the aid of the present rotation factors would then be a certain mean of velocities derived from different layers which may either exceed somewhat, or be smaller than, the true velocity at the equator—depending on the geometry of each particular eclipse. On the other hand, provided that the law of increase of the angular velocity of rotation with diminishing latitude were known, it could be incorporated in the formulae for F, and very accurate observations might then enable us to obtain some information concerning the variation of angular velocity with latitude for stars other than the sun.

A further assumption underlying the present derivation of F is that the axis of rotation of the eclipsed star is constantly perpendicular to the orbital plane. In certain cases the validity of such an assumption may be problematic; for if the tides raised by the eclipsing star in the surface layers of its massive mate are small and if the latter are of minute density, no physical reason is known which should impel the massive star to keep its axis of rotation exactly perpendicular to the orbital plane. Moreover, a deviation from perpendicularity would entail no observational consequences that could in practice be detected otherwise than by the study of

the rotational effect. That is, if the axis of rotation is not perpendicular to the orbital plane, the rotational effect should no longer be symmetrical with respect to conjunction, and need not be zero at mid-primary minimum. The asymmetry of the rotational effect should also vanish and reappear periodically as the apparent inclination of the axis of rotation oscillates during the precessional motion. Such a situation may arise in systems where the component undergoing eclipse is very small or very massive compared with the eclipsing star. It seems indicated by observations, for instance, in the case of Algol;[8] but its discussion would widely exceed the scope of the present paper and is postponed for a future communication.

Summary.—Analytical formulae are set up for rigorous computation of the rotation factors required for the interpretation of the spectroscopically observed rotational effect in eclipsing binary systems. The formulae are exact as long as the components rotate like rigid bodies and their axes of rotation are perpendicular to the orbital plane. No notice will be taken at present of complications arising from the distortion of the components in form or from orbital eccentricity; and the components are assumed to appear as uniformly bright or limb-darkened circular discs. The rotation factors are found to be expressible in terms of the same functions that are required for an analysis of light changes of eclipsing variables. All such functions can be made to depend on two independent parameters—say, the ratio of radii k and the geometrical depth of the eclipse p—and tabulated in terms of these parameters for any particular phase of the eclipse. Tables already available for two such functions should greatly facilitate the computation of the rotation factors in practical cases.

[1] *Publ. D. A. O.*, **7**, 133 (1938).

[2] *Proc. Am. Phil. Soc.*, in press.

[3] Provided that the rotation is direct. No case of retrograde rotation among eclipsing binaries has been detected so far.

[4] Analytical expressions for uniform discs have already been set up by Petrie (op. cit.). His formulae are, however, unnecessarily complicated. If we verify that, in Petrie's notations,

$$\sqrt{1 - y_1^2} + \sqrt{k^2 - y_1^2} = \delta,$$

his formulae become identical with those given presently.

[5] *Bull. Inst. Astr. Leningrad*, No. 45 (1939), and No. 50 (1940).

[6] The present formulae should then be in error by a factor of the order of r_3^4.

[7] Cf. Jeans, *Astronomy and Cosmogony*, Cambridge, 1928, Chapter X.

[8] Cf. McLaughlin, *Michigan Publ.*, **6**, 3 (1934).

CERTAIN DIRECT PRODUCTS OF THE GROUPS OF SELF-ISOMETRIES

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The groups of rotations of the regular polygons are commonly called the dihedral groups. A necessary and sufficient condition that a dihedral group is non-abelian is that its order exceeds 4. If such a non-abelian group is the direct product of two of its proper subgroups one of these subgroups contains at most two invariant operators. If one of the two given factor groups contains two invariant operators the other contains no invariant operator besides the identity and hence it is a dihedral group whose order is twice an odd number. The former factor group is then of order 2. That is, *a necessary and sufficient condition that a non-abelian dihedral group is the direct product of two of its proper subgroups is that its order is divisible by 4 but not by a higher power of 2.* The only abelian dihedral group is the four group and hence it is the direct product of every two of its three proper subgroups.

The group of rotations of a regular polygon is also its group of self-isometries. That is, it is the most general group which transforms the regular polygon into itself in such a way that every pair of its points is transformed into a pair of points which are at the same distance from each other as the original points were. This results from the fact that such a group leaves the center of the polygon fixed and transforms its vertices transitively. If such a vertex is fixed another vertex can be transformed in two ways so that the order of the group of its self-isometries is twice the number of its vertices. If the number of the vertices is even a reflection of the polygon on its center corresponds to an invariant operator of order 2 in its group of self-isometries, but this group is the direct product of the subgroup generated by this invariant operator of order 2, which corresponds also to the rotation of the regular polygon in its plane through an angle of 180°, and some other proper subgroup of the group of self-isometries of the polygon if, and only if, the order of this group is divisible by 4 but by no higher power of 2, as was noted above.

The groups of movements of the five regular polyhedrons have received much attention and some of their abstract definitions were noted already by W. R. Hamilton (1805–1865); cf. *Bibliotheca Mathematica*, volume 10, page 314 (1910–1911). The groups of self-isometries of these polyhedrons have also been noted. For instance, on page 342 of volume 1 (1926) of his *Entwicklung der Mathematik*, Felix Klein refers to the group of self-isometries of the regular icosahedron, which is of order 120, and notes that it is

not the same as the symmetric group of degree 5 since it contains an invariant operator of order 2, but he does not point out here that it is the direct product of the group of movements of the regular icosahedron and a group of order 2. This fact is of fundamental importance because *all the groups of self-isometries of the five regular polyhedrons are the direct products of an invariant subgroup of order* 2 *contained therein and the corresponding group of movements of the regular solid*, and their abstract definitions can easily be obtained from the known abstract definitions of the groups of these solids.

If a group contains an invariant operator of order 2 and a subgroup of index 2 which does not include this invariant operator then it is the direct product of two proper subgroups of which one is the subgroup of order 2 generated by this invariant operator. From this obvious theorem it results directly that each of the groups of the self-isometries of the five regular solids is the direct product of two proper subgroups, one of these is the group of movements of the solid while the other is the group of order 2 generated by the operator of order 2 representing the reflections on the center of the regular solid, as was noted in the preceding paragraph. As the properties of such direct products are readily obtained from those of the given subgroup of index 2, it results that the groups of the self-isometries of the five regular solids present little which is of interest from the point of view of group theory beyond that of the groups of movements of these solids.

Since the group of movements of the cube is the same as the group of the regular octahedron and the group of the regular icosahedron is the same as the group of the regular duodecahedron there are three distinct groups of movements of the five regular solids. These are generated by two operators of orders 2 and 3 whose product has an order which is equal to one of the three numbers 3, 4, 5, respectively. From these relations it is easy to obtain abstract definitions of the three groups of self-isometries of the regular solids. In the case of the regular tetrahedron we may say that the group of its self-isometries is generated by two operators of orders 2 and 3, respectively, which satisfy the condition that the product is an operator of order 6 whose cube is invariant under the group. In fact, if s_1 and s_2 are these two generating operators of orders 2 and 3, respectively, we may observe that

$$1,\ s_1i,\ s_2{}^2s_1s_2i,\ s_1s_2{}^2s_1s_2$$

is the four group and that this is invariant under the group generated by s_1 and s_2.

Since s_1i and $s_2{}^2s_1s_2i$ are both of order 2 they are commutative if their product is of this order. That this is the case results from the fact that $(s_1s_2)^3$ is equal to i. The given four group is invariant under s_1 since s_1 transforms each of the last two of the given operators into each other. It is

invariant under s_2 because s_2 transforms its three operators of order 2 cyclically. Hence the given four group and s_2 generates the tetrahedral group which is invariant under i and the given relations constitute an abstract definition of the group of self-isometries of the regular tetrahedron. The quotient group of this group with respect to the invariant subgroup of order 2 is the tetrahedral group, and the operator of order 2 in the given defining relations had to be so chosen that it does not appear in the tetrahedral subgroup because the given operator of order 3 necessarily appears in this group.

A set of defining relations for the abstract group of order 48 which is simply isomorphic with the group of self-isometries of the cube is that its two generators of orders 2 and 4, respectively, have a product of order 6 whose cube i is invariant under this group of order 48. From $s_1^2 = s_2^4 = (s_1s_2)^6 = 1$ and $(s_1s_2)^3 = i$ it is easily seen that the group generated by s_1 and s_2 contains as an invariant subgroup the four group defined by

$$1,\ s_2^2, s_1s_2^2, s_2^2s_1$$

The second and third of these four operators are of order 2 by definition and the fact that the order of a transform of an operator by some other operator is the same as the order of the original operator. The order of the fourth of these operators is 2 as a result of the equation $(s_1s_2)^3 = i$. Hence the three given operators of order 2 are commutative and this four group is invariant under the entire group of order 48.

Since $(s_1s_2)^2$ is of order 3 and transforms the three given operators of order 2 cyclically it and this four group generate the tetrahedral group. This tetrahedral group contains s_1s_2i as well as $(s_1s_2)^2$, and includes all the operators of odd order which appear in the given group of order 48. This contains two subgroups which are simply isomorphic with the symmetric group of order 24. One of these is obtained by extending the given tetrahedral group by s_1 while the other is obtained by extending this subgroup by means of the operator s_1i. This group is simply isomorphic with the transitive permutation group of degree 6 and of order 48, which has received much attention in the theory of permutation groups in the early development of this subject. It represents the interchanges of the six plane surfaces of the square under its group of self-isometries.

A set of independent generators of the abstract group which is simply isomorphic with the group of self-isometries of the regular icosahedron can be selected in various ways. In the first place we may consider the abstract group which is generated by the two operators s_1 and s_2 of orders 2 and 3, respectively, which satisfy the condition that their product is of order 10 and that the fifth power of this product is an invariant operator of order 2. From $(s_1s_2)^5 = i$ it results that s_1s_2i is an operator of order 5 and that s_1i, s_2 are two operators of orders 2 and 3, respectively, whose product is of

order 5. It therefore results that they generate the icosahedral group as was noted already by W. R. Hamilton. Since i does not appear in this icosahedral group it follows that the group of isometries of the regular icosahedron is simply isomorphic with the direct product of the icosahedron group and the group of order 2, as was noted above. An abstract definition of this direct product is similarly seen to be the group generated by an operator of order 2 and an operator of order 5 whose product is of order 6, but has a cube which is commutative with these two operators.

CERTAIN CONGRUENCE CRITERIA CONNECTED WITH FERMAT'S LAST THEOREM

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In another paper[1] the writer obtained a result which may be stated as follows:

Theorem I. *If l is an odd prime and*

$$x^l + y^l + z^l = 0 \tag{1}$$

and $y \equiv 0 (\mathrm{mod}\ l)$; $(xz, l) = 1$, the prime ideal $\mathfrak{p}$ in $k(\zeta)$, with $\zeta = e^{2i\pi/l}$ is a divisor of the rational prime p;

$$c = \frac{N(\mathfrak{p}) - 1}{l}$$

where $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$, further, $\mathfrak{p}$ belongs to an exponent[2] which is prime to l with $xyz \not\equiv 0 (\mathrm{mod}\ p)$, then

$$\binom{c}{s} + \binom{c}{s+l} t^l + \binom{c}{s+2l} t^{2l} + \cdots \tag{2}$$

is divisible by p for $s = 1, 2, \ldots, l-1$, and $y/x = t$.

The present article will be devoted to extensions of the methods which were employed in the proof of this theorem. A related method is employed in the proof required for the demonstration of a theorem in another article.[3] A companion formula to (2) which is obtained by the method[1] we previously referred to is

$$\frac{1}{x^c} \equiv \binom{c}{0} + \binom{c}{l} t^l + \binom{c}{2l} t^{2l} + \cdots \tag{3}$$

The condition $xyz \not\equiv 0 \pmod{p}$ is of course, satisfied if we take p sufficiently large in (2) and (3). Also, we note that in view of (1) we can employ z in place of x in our argument, hence (2) holds for $t = y/x$ and $t_1 = y/z$. For p sufficiently large these are incongruent modulo p. Hence (2) holds for two incongruent values of t.

Applying our second method[3] we obtain from (1), under the assumption that the second factor of the class number of the field defined by ζ is prime to l, the congruence

$$(x + \zeta^a y)^c \equiv (x + \zeta^{-a} y)^c \pmod{\mathfrak{p}} \qquad a = 0, 1, 2, \ldots, l - 1 \quad (4)$$

This gives[4]

$$\binom{c}{s} x^s y^{c-s} + \binom{c}{s+l} x^{s+l} y^{c-(s+l)} + \ldots$$

$$\equiv \binom{c}{l-s} x^{l-s} y^{c-(l-s)} + \binom{c}{2l-s} x^{2l-s} y^{c-(2l-s)} + \ldots \quad (5)$$

and we may state the

THEOREM II. *If (1) is satisfied under the conditions given in Theorem I and the second factor of the class number of $k(\zeta)$ is prime to l, then for each prime p exceeding a certain limit the congruences (5) hold, and they also hold when x is replaced by z; $xyz \not\equiv 0$.*

Elsewhere[3, 4] the congruences (5) were employed in a bit different way. There, p was in the particular form $1 + cl$ and we considered smaller values for c. Employing this idea we were able to obtain criteria which led to the proof of Fermat's last Theorem for all prime exponents < 619. We also find[5] from (1)

$$(x + \zeta^k z)^c \equiv (\zeta^k x + z)^c \pmod{\mathfrak{p}} \quad (6)$$

$$k = 0, 1, 2, \ldots l - 1$$

which gives

$$\binom{c}{s} x^s y^{c-s} + \binom{c}{s+l} x^{s+l} y^{c-(s+l)} + \ldots$$

$$\equiv \binom{c}{s} z^s x^{c-s} + \binom{c}{s+l} z^{s+l} x^{c-(s+l)} + \ldots. \quad (7)$$

which provides another criterion for the solution of (1) in addition to Theorem II.

Much work has been done on Fermat's Last Theorem in connection with the congruence

$$x^l + y^l + z^l \equiv 0 \pmod{p} \quad (8)$$

where $p \neq l$. It is known[6] that if p exceeds a certain limit this congruence always has solutions with x, y and z, prime to p. This may also be true in connection with the congruences (5) and (7) but it does not appear obvious why this is so. (5) is obviously satisfied when $y \equiv 0 (\mathrm{mod}\ p)$ but p can be taken $> y$. Similarly (7) is satisfied if either $x + z$, or $x - z$ is divisible by p but again p can be taken $> x + z$ or $x - z$. For the smaller values of c, the only solution of (5) is $y \equiv 0 (\mathrm{mod}\ p)$.

In the statement of Theorem I, $\mathfrak{p}$ was limited to primes belonging to an exponent which was prime to l. Now the writer has shown[7] that for there to exist an ideal not of the first degree in the basis classes C of the irregular class group of $k(\zeta)$ defined by

$$C^{s - r^l - 2a} = 1, \tag{8a}$$

r being a primitive root of l, it is necessary that $l - 2a$ and $l - 1$ have a common factor, >1. It will be found that this condition is also necessary when we take classes C_1 of the irregular class group of $k(\zeta)$ defined by

$$C_1^{s - r^{2a}} = 1.$$

If we apply this necessary condition to all the fields $k(\zeta)$ for <619 we find that there are only eight values for l in this range where it is possible for these types of classes to contain other than ideals of the first degree. If it is true that all of these irregular classes contain only ideals of the first degree, then Theorem I may be greatly simplified, as we need only state that p in (2) belongs to some exponent > 1 modulo l.

From the congruences (2) we may now obtain congruences independent of t. For, we may take (2) for a particular value of s and multiply this congruence by

$$t^l, t^{2l} \ldots, t^{l(c-1)}; \tag{9}$$

in this way we obtain a set of c congruences, which yield the relation

$$D \equiv 0 (\mathrm{mod}\ p), \tag{10}$$

where D is the circulant

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots . 0 \\ 0 & a_0 & a_1 & a_2 \ldots 0 \\ 0 & 0 & a_0 & a_1 \ldots 0 \\ \ldots\ldots\ldots\ldots\ldots \\ a_1 & a_2 & a_3 & a_4 \ldots a_0 \end{vmatrix} \tag{11}$$

and where, if $i = 1, 2, \ldots, h$,

$$a_i = \binom{c}{s + il}$$

$$h = \left[\frac{c}{l}\right],$$

$[u]$ being the greatest integer in u. Now we may obtain a simpler determinant when $p^f \equiv 1 \pmod{l^2}$ and $f > 1$ with $c_1 l \equiv c$, for if we multiply the congruences (2) by

$$t^l, t^{2l}, \ldots, t^{l(c_1 - 1)}$$

we obtain a set of congruences from which we may infer, if we note that since $f > 1$, $t^{c'} \equiv 1 \pmod{p}$; $D_1 \equiv 0 \pmod{p}$, where

$$D_1 = \begin{vmatrix} a_0 & a_1 \ldots a_{c_1 - 1} \\ a_{c_1 - 1} & a_0 \ldots a_{c_1 - 2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_1 & a_2 \ldots a_0 \end{vmatrix} \tag{11a}$$

for $s > 0$.

If we now take (2) again for a fixed s and multiply this congruence by each of the quantities in (10), and consider any $(c - 1)$ of the resulting congruences as a set of linear congruences in the quantities in (9), and employing the method used in deriving Cramer's rule, we find that

$$\Delta t^l \equiv A \pmod{p} \tag{12}$$

where Δ is now the determinant formed by the coefficients of the quantities in (9), and where (A) is obtained from the determinant Δ by replacing the elements in the first column by the elements in the congruences which do not involve t. Now (12) is a congruence in t which is satisfied by each of the quantities

$$t, \zeta t, \zeta^2 t, \ldots, \zeta^{l-1} t.$$

Now consider the modulus $\mathfrak{p}$, a prime ideal divisor of p, and take in addition t_1 in place of t, in (12). The relation $t_1 \equiv t\zeta^b \pmod{p}$ gives $t^l - t_1^l \equiv 0 \pmod{p}$, and this is impossible for p sufficiently large. Then, (12) with $\mathfrak{p}$ in place of p has more than l incongruent solutions unless $\Delta \equiv 0 \pmod{p}$ in which case $A \equiv 0 \pmod{p}$. If we also consider the relation

$$\Delta t^{kl} \equiv A_k \pmod{p}$$

obtained from our former congruences, it will therefore be seen that

$$A_k \equiv 0 \pmod{p}.$$

By taking $s = 1, 2, \ldots l - 1$, we obtain in this way a number of determinants each divisible by p. If $p^f \equiv 1 \pmod{l^2}$ we obtain as in the derivation of (11a) a set of simpler determinant criteria.

We now apply methods similar to those used in the derivation of Theorem I to the first case of Fermat's Last Theorem. In (1) we now assume that

$$(xyz, l) = 1$$

and we select a prime ideal $\mathfrak{p}$ in $k(\zeta)$ such that if $\lambda = (1 - \zeta)$,

$$\mathfrak{p} = (1 + m\lambda^l + \theta\lambda^{l+1}) = (\omega) \tag{13}$$

where m is a rational integer $\not\equiv 0 \pmod{l}$. By a known theorem[8] we may find an infinity of θ's, if m is fixed, so that an expression of this form gives an infinity of prime ideals. Since ω is primary, we have[9]

$$\left(\frac{x + y\zeta^s}{\omega}\right) = \left(\frac{\omega}{x + y\zeta^s}\right) = 1 \tag{14}$$

The relation (13) shows that $\mathfrak{p}$ is an ideal of the first degree, hence

$$p = 1 + cl.$$

Then we may write, using (14), and if $xy \not\equiv 0 \pmod{p}$,

$$(x + y\zeta^s)^c \equiv 1 \pmod{\mathfrak{p}} \tag{15}$$

where $s = 0, 1, \ldots, l - 1$, if we assume that $x + \zeta^s y \not\equiv 0 \pmod{p}$. On expanding the left hand member of this congruence and using the various values of s we obtain congruences of the same type as (2) for $s = 1, 2, \ldots, l - 1$, and similarly we find that a determinant of the same type as (11) is divisible by p. In the discussion of the second case of our theorem, we took values of p large enough so that various congruences involving x, y and z did not hold. However, for the first case, which we are now discussing, this is not necessary. For, if $x + \zeta^a y \equiv 0 \pmod{\mathfrak{p}}$, then since $\not\equiv 0 \pmod{p}$, $x + \zeta^s y \not\equiv 0 \pmod{p}$ for $s \not\equiv a \pmod{p}$. Under this condition (5) gives

$$x^c(\zeta^a - \zeta^s)^c \equiv 1 \pmod{\mathfrak{p}} \tag{16}$$

and (1) gives

$$x + z = v^l$$

yielding

$$x \equiv v^l \pmod{p}$$

or

$$x^c \equiv 1 \pmod{p}$$

hence from (16) we have

$$(\zeta^a - \zeta^s)^c \equiv 1 (\text{mod } \mathfrak{p})$$

but by a known result[10] this is impossible since $m \not\equiv 0 (\text{mod } l)$ in (13).

We may then state the

THEOREM III. *Let l be an odd prime,*

$$x^l + y^l + z^l = 0$$

and $y \equiv 0 (\text{mod } l)$; $(xz, l) = 1$; the prime ideal $\mathfrak{p}$ is a divisor, in the field $k(\zeta)$; $\zeta = e^{2i\pi/l}$, of the rational prime p;

$$c = \frac{N(\mathfrak{p}) - 1}{l}$$

where $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$. Further, $\mathfrak{p}$ belongs to an exponent which is prime to l; then for any p sufficiently large the determinant (11) is divisible by p, with $i = 1, 2, \ldots, h$,

$$a_i = \binom{c}{s + il}$$

$$h = \left[\frac{c}{l}\right],$$

for $s = 1, 2, \ldots l - 1$. Also $[u]$ is the greatest integer in u. In addition, certain minors of (11), as indicated in the relation (12), are divisible by p. In particular, if p belongs to an exponent > 1, and $p^j = 1 (\text{mod } l^2)$ the determinant (11a) is divisible by p for each s mentioned above.

We also have

THEOREM IV. *Let l be an odd prime,*

$$x^l + y^l + z^l = 0$$

with $(xyz, l) = 1$, and $\mathfrak{p}$ is a prime ideal in $k(\zeta)$ such that

$$\mathfrak{p} = (1 + m\lambda^l + \theta\lambda^{l+1})$$

where $m \not\equiv 0 (\text{mod } l)$. Further set $N(\mathfrak{p}) = p$ with c defined as in Theorem II. Then the determinant (11) is divisible by p, for $s = 1, 2, \ldots, l - 1$, a_i and h being defined as in Theorem II. Also certain minors of (11) as indicated in the relations (12) are divisible by p.

The statements of Theorems I and III each include the condition "$\mathfrak{p}$ belongs to an exponent which is prime to l;" we have, however, as already indicated no convenient criteria for determining if a given prime ideal satisfies such a condition. We shall now point out, however, that there are certain classes of ideals which may be immediately recognized as belonging

to an exponent prime to l. For if p belongs to the exponent $f(\text{mod } l)$ and such that $ef = l - 1$ with e odd, and if r is a primitive root of l,

$$\mathfrak{p}(\zeta) = \mathfrak{p}(\zeta^{r^e})$$

which gives from (8a)

$$r^{e(l-2a)} \equiv 1(\text{mod } l)$$

which is impossible since r is a primitive root of l and the exponent is odd.

[1] *Proc. Nat. Acad. Sci.*, **15**, 45 (1929).

[2] We say that an ideal $\mathfrak{p}$ belongs to an exponent n when $\mathfrak{p}^n$ is a principle ideal but $\mathfrak{p}^d$ is not a principal ideal for $0 < d < n$.

[3] *Trans. Amer. Math. Soc.*, **31**, 632 (1929).

[4] *Duke Math. Jour.*, **5**, 419–420 (1939).

[5] *Trans. Amer. Math. Soc.*, **31**, 632–633 (1929).

[6] Dickson, *Jour. für. Math.*, **135**, 134–141 (1909).

[7] *Proc. Nat. Acad. Sci.*, **16**, 303 (1930).

[8] Landau, *Math. Zeitschrift*, **2**, 52–154 (1918).

[9] Hasse, *Jahresbericht Deutschen Math.-Verein*, Ergänzungsbände VI, 61 (1930).

[10] Hasse, loc. cit., p. 112.

ON THE PHYSICAL CHARACTERISTICS OF THE HYDRA CLUSTER OF NEBULAE

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A. Observational Data.—In a previous paper[1] counts of the brighter nebulae in the Hydra cluster were communicated. The distance of the Hydra cluster was estimated as 7.3×10^6 parsecs and the average apparent velocity of recession to be expected was given as $v = 4100$ km./sec. No observations of the red shift for nebulae in the Hydra cluster were available at the time. Dr. Hubble subsequently obtained a spectrum plate for NGC 3309 which nebula is a member of the Hydra cluster. He kindly informs me that the apparent velocity of recession for NGC 3309 is of the order of 3950 km./sec., a value which is in good agreement with our original estimate.

B. The Radial Distribution of Nebulae in the Hydra Cluster.—As emphasized previously[1] the Hydra cluster exhibits spherical symmetry and is therefore suspected to have reached a statistically stationary state. In order to check this conclusion two additional tests are available. In the

first place we may compare the observed radial distribution of nebulae in the Hydra cluster with the distribution derived by Emden[2] for the bounded isothermal gravitational gas sphere and secondly we can verify whether the ratio $\overline{w_r^2}/\rho_0$ of the square of the dispersion in radial velocities of the cluster nebulae to the central density ρ_0 of the cluster can be correctly determined from the observed *structural length or structural index* α of the cluster.[3]

If from the previously[1] given counts in the Hydra cluster of nebulae brighter than the apparent photographic magnitude $m_p = 16.2$ we subtract the nebulae which belong to the general background (1.8 nebulae per square degree), we obtain for the average number of nebulae N_r per square degree at various distances r from the center of the cluster the values given in table 1.

TABLE 1

RADIAL DISTRIBUTION OF NEBULAE IN THE HYDRA CLUSTER

r IN MINUTES OF ARC	N_r	8.2 N_r	r_1	1000 D	1000 D − 37
0	...	....	0	3032	2995
10	241.8	1983	2.5	1815	1778
20	121.2	994	5	832	795
30	44.6	366	7.5	476	439
40	24.9	204	10	318	281
50	30.7	252	12.5	243	206
60	24.9	204	15	195	158
70	13.3	109	17.5	163	126
80	11.6	95	20	140	103
90	7.7	63	22.5	126	89
100	1.9	15.6	25	111	74
110	6.4	52.5	27.5	97	60
120	5.3	43.5	30	92	55
130	1.0	8.2	32.5	87	50
140	4.2	34.4	35	81	44
150	5.4	44.3	37.5	75	38
160	...	....	40	69.6	32.6
170	2.4	19.7	42.5	66.4	29.4
180	2.1	17.2	45	63.1	26.1
190	2.6	21.3	47.5	59.8	22.8
200	2.4	19.7	50	56.5	19.5
210	1.3	9.1	52.5	54.6	17.6
220	0.6	4.9	55	52.7	15.7
230	2.6	21.3	57.5	50.8	13.8
240	1.3	9.1	60	49	12
250	0.5	4.6	62.5	45	8
280	...	....	70	41.5	4.5
340	...	....	85	37	0

Column 4 gives the distances r_1 in the standard[3] Emden isothermal gas sphere to which the actual distances r from the center of the cluster are

reduced through the relation $r = \alpha r_1$. The values of D represent the spatial densities in an infinite isothermal sphere while $D - 37/1000$ is the projected density in that bounded isothermal sphere which best

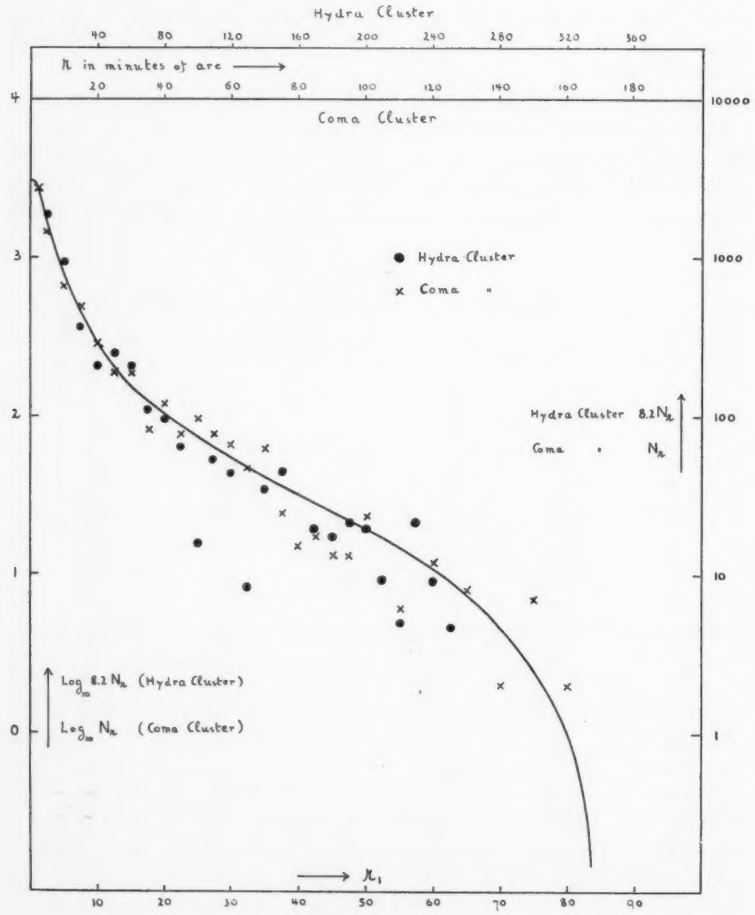

FIGURE 1

represents the radial distribution of the brighter nebulae in the Coma cluster with which we here compare the data on the Hydra cluster. From figure 1 we see that the properly reduced radial distributions of the Coma cluster and of the Hydra cluster agree among themselves, as well as with

the theoretical curve, within the limits of statistical fluctuations to be expected for the relatively small number of nebulae involved in the counts.

There is some indication that the values of N_r for the Hydra cluster begin to fall off from the theoretical curve at slightly smaller values of r_1 than those for the Coma cluster. This checks the general expectation that, the smaller the population of a cluster (and the higher its velocity dispersion) the smaller should be the distance from the center of the cluster at which the Boltzmann distribution changes over into a Smoluchowski distribution.

The data for the Coma cluster plotted in figure 1 include the nebulae brighter than the apparent magnitude $m_p = 16.6$ or brighter than the absolute magnitude $M_p = -14.5$ while the data for the Hydra cluster include the nebulae brighter than the apparent magnitude $m_p = 16.2$ or the absolute magnitude $M_p = -13.1$.

C. Physical Characteristics of the Hydra Cluster.—From figure 1 we deduce that the structural length $\alpha = r/r_1$ for the Coma and Hydra clusters in angular measure are 2′ arc and 4′ arc, respectively. Since the distances of the two clusters are 13.8×10^6 and 7.3×10^6 parsecs, respectively, we obtain for the

Coma Cluster: $\alpha = 2.48 \times 10^{22}$ cm. (1)

Hydra Cluster: $\alpha = 2.56 \times 10^{22}$ cm. (2)

By definition[3] it is

$$\alpha = (\overline{w^2}/12\pi\Gamma\rho_0)^{1/2} \tag{3}$$

where $\overline{w^2}$ is the square of the velocity dispersion in a stationary cluster and ρ_0 is its central density. Since the velocity dispersion for the nebulae in the Hydra cluster is not known at the present we give in table 2 values for ρ_0 in dependence of a number of values for the velocity dispersion.

TABLE 2

$(\overline{w^2})^{1/2}$ IN KM./SEC.	ρ_0 IN G./CM.^{3}	$\overline{\mathfrak{M}}/\mathfrak{M}_\odot$
250	3.8×10^{-25}	5.8×10^9
500	1.5×10^{-24}	2.3×10^{10}
750	3.4×10^{-24}	5.2×10^{10}
1000	6.0×10^{-24}	9.2×10^{10}
1500	1.4×10^{-23}	2.1×10^{11}
2000	2.4×10^{-23}	3.7×10^{11}
2500	3.8×10^{-23}	5.8×10^{11}

In column 3 are given the ratios of the average mass $\overline{\mathfrak{M}}$ of the nebulae involved to the mass $\mathfrak{M}_\odot$ of the sun which correspond to the different values of the velocity dispersion given in column 1. The method of calculation

of the ratios $\overline{\mathfrak{M}}/\mathfrak{M}_\odot$ is given in another place[2] where it is shown that for the same values of $(\overline{w^2})^{1/2}$ the ratios $\overline{\mathfrak{M}}/\mathfrak{M}_\odot$ for the Coma cluster are about half as large as those here obtained for the Hydra cluster. Now the nebulae included in the counts of the Coma cluster are brighter than the absolute magnitude $M_p = -14.5$ while those for the Hydra cluster are brighter than $M_p = -13.1$. Therefore if the same types of nebulae are included in the two clusters $\overline{\mathfrak{M}}$ (Coma) should be slightly larger than $\overline{\mathfrak{M}}$ (Hydra). This means that $2\overline{w^2}$ (Hydra) should be slightly smaller than $\overline{w^2}$ (Coma) or the velocity dispersion in the Hydra cluster should be smaller than that in the Coma cluster by a factor slightly smaller than 0.7. It will be interesting to check this conclusion by an investigation of the velocity dispersion in the Hydra cluster.

From our data we can also answer the question regarding the central density, that is the number of nebulae, say, per cubic mega-parsec (10^{18} cubic parsecs) in the two clusters. It was shown in another place[3] that

$$\rho_0 = \sigma_0/3.03\alpha \qquad (4)$$

where ρ_0 is either the central spatial density or the number of nebulae per unit cube while σ_0 is the projected central density. From the extrapolation of our counts of nebulae to the center of the two clusters we consequently obtain the following results.

In the center of the Hydra cluster we obtain per cubic mega-parsec 8.7×10^5 nebulae which are brighter than the absolute magnitude $M = -13.1$ while the corresponding number for the Coma cluster is 2×10^6 nebulae per cubic mega-parsec which are brighter than the absolute magnitude $M = -14.5$. In comparison we mention that in the general field the average number of nebulae in the same range of absolute magnitudes is less than ten per cubic mega-parsec. The number of nebulae per unit volume in the center of the large clusters is therefore considerably higher than even experts in nebular counts might have expected. The high nebular density in clusters perhaps sheds some light on the remarkable fact that large symmetrical clusters of nebulae show all of the physical characteristics required for statistically stationary assemblies of objects whose interactions are governed by Newton's law. The peculiar possibility that these giant clusters whose elementary building stones are the nebulae themselves should provide the first quantitative test for Emden's results which were derived for gaseous spheres whose elementary building stones are atoms or molecules requires of course additional and thorough investigation.

In conclusion we mention that the clusters in Perseus, Cancer, Fornax and Pegasus as well as some of the groups in Pisces lend themselves to the

same analysis as that applied here to the Hydra cluster and in another place to the Coma cluster.[3]

[1] Zwicky, F., these PROCEEDINGS, **27,** 264 (1941).

[2] Emden, R., *Gaskugeln*, Teubner, Leipzig, 1907.

[3] Zwicky, F., *Th. von Kármán Anniversary Volume*, May, 1941. See also a paper on the clustering of nebulae which is to appear shortly in the *Astrophys. Journal.*